

# Solution of polynomiality and positivity constraints on generalized parton distributions

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An integral representation for generalized parton distributions of spin-0 hadrons is suggested, which satisfies both polynomiality and positivity constraints.

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## I. INTRODUCTION

Generalized parton distributions (GPDs) [1–10] appear in the context of the QCD factorization in various hard exclusive phenomena including deeply virtual Compton scattering and hard exclusive meson production. Among several general constraints on GPDs an important role is played by the polynomiality of the Mellin moments [5] and by the positivity bounds [11–22]. In this paper we suggest a representation for GPDs which automatically satisfies both positivity and polynomiality constraints. The analysis is restricted to the case of spin-0 hadrons (e.g. pions) but various types of partons will be covered.

We use the following definition of GPDs:

$$H^{(N)}(x, \xi, t) = \int \frac{d\lambda}{2\pi} \exp(i\lambda x) \langle P_2 | O^{(N)}(\lambda, n) | P_1 \rangle. \quad (1)$$

Here  $|P_k\rangle$  is the hadron state with momentum  $P_k$ . The light-like vector  $n$ ,

$$n^2 = 0,$$

is normalized by the condition

$$n(P_1 + P_2) = 2. \quad (2)$$

We use the standard notation of Ji [14] for the parameters  $\Delta$ ,  $t$  and  $\xi$ :

$$\Delta = P_2 - P_1, \quad \xi = -\frac{1}{2}(n\Delta), \quad t = \Delta^2. \quad (3)$$

The definitions of light-ray operators  $O^{(N)}(\lambda, n)$  for various types of partons are listed in the Table I. We have included the scalar field  $\phi$  into this table since the positivity bounds are more general than their applications in QCD. The last column of this table contains the number  $N$  of factors  $n^\mu$  appearing in the light-ray operator  $O^{(N)}(\lambda, n)$ . This number  $N$  plays an important role in the formulation of the positivity bounds and of the polynomiality conditions and we include  $N$  in the notation (1) of GPD  $H^{(N)}(x, \xi, t)$ .

The structure of the paper is as follows. Section II contains a brief description of the polynomiality and positivity constraints on GPDs. In Sec. III we introduce a modified version of the double distribution representation for GPDs which slightly differs from the standard one but is more relevant for our aims. In Sec. IV an ansatz for the double dis-

tribution is suggested. In the remaining part of the paper we show that this ansatz leads to GPDs which obey both polynomiality and positivity constraints. In Sec. V we compute the GPDs corresponding to our ansatz for double distributions. Section VI contains the check of the positivity of the forward parton distributions. The positivity bounds on GPDs are verified in Sec. VII. The Appendix contains the derivation of the general solution of the positivity bounds (without the polynomiality constraints).

## II. POLYNOMIALITY AND POSITIVITY

Whatever limited our knowledge about GPDs is, there are two basic constraints: polynomiality and positivity. The polynomiality means that Mellin moments in  $x$  of GPD  $H^{(N)}(x, \xi, t)$

$$\int_{-1}^1 dx x^m H^{(N)}(x, \xi, t) = P_{m+N}(\xi, t) \quad (4)$$

must be polynomials in  $\xi$  of degree  $m + N$ .

Various inequalities for GPDs suggested in the Refs. [11–21] can be considered as particular cases of the general positivity bound on GPDs derived in Ref. [22]. This general positivity bound has a relatively simple formulation in the impact parameter representation [16,19–21,23]. The impact parameter  $b^\perp$  appears via the Fourier transformation of the  $\Delta^\perp$  dependence of GPDs. If the transverse plane is orthogonal to vectors  $n$  and  $P_1 + P_2$ , then the transverse component  $\Delta^\perp$  of the momentum transfer  $\Delta$  (3) is connected with the variable  $t = \Delta^2$  by the relation

TABLE I. Light-ray operators  $O^{(N)}(\lambda, n)$  for various types of partons and the corresponding parameter  $N$ .

Parton	$O^{(N)}(\lambda, n)$	$N$
scalar	$\phi^* \left( -\frac{\lambda n}{2} \right) \phi \left( \frac{\lambda n}{2} \right)$	0
quark	$\frac{1}{2} \bar{\psi} \left( -\frac{\lambda n}{2} \right) (n \cdot \gamma) \psi \left( \frac{\lambda n}{2} \right)$	1
gluon	$n^\mu G_{\mu\nu}^a \left( -\frac{\lambda n}{2} \right) n_\rho G^{a,\nu\rho} \left( \frac{\lambda n}{2} \right)$	2

$$t = -\frac{|\Delta^\perp|^2 + 4\xi^2 M^2}{1 - \xi^2}. \quad (5)$$

Here  $M$  is the mass of the hadron. We define the GPD in the impact parameter representation as follows:

$$\begin{aligned} \tilde{F}^{(N)}(x, \xi, b^\perp) &= \int \frac{d^2 \Delta^\perp}{(2\pi)^2} \exp[i(\Delta^\perp b^\perp)] \\ &\times H^{(N)}\left(x, \xi, -\frac{|\Delta^\perp|^2 + 4\xi^2 M^2}{1 - \xi^2}\right). \end{aligned} \quad (6)$$

Here notation  $\tilde{F}^{(N)}$  is used in order to avoid confusion with the nucleon GPD  $\tilde{H}$  and to keep the compatibility with the notation of Ref. [22], where the following inequality was derived:

$$\begin{aligned} \int_{-1}^1 d\xi \int_{|\xi|}^1 dx (1-x)^{-N-4} p^*\left(\frac{1-x}{1-\xi}\right) p\left(\frac{1-x}{1+\xi}\right) \\ \times \tilde{F}^{(N)}\left(x, \xi, \frac{1-x}{1-\xi^2} b^\perp\right) \geq 0. \end{aligned} \quad (7)$$

This inequality was obtained in Ref. [22] for the case  $N=1$  and the generalization to arbitrary  $N$  is straightforward.

Inequality (7) should hold for any function  $p(z)$ . Therefore we actually deal with an infinite set of positivity bounds on the GPD. The general inequality (7) covers various inequalities suggested for GPDs [11–21] as particular cases with some special choice of functions  $p(z)$ .

It is well known that the double distribution representation [1,2,7] with the  $D$ -term [24]

$$\begin{aligned} H(x, \xi, t) &= \int_{|\alpha|+|\beta|\leq 1} d\alpha d\beta \delta(x - \xi\alpha - \beta) \bar{F}_D(\alpha, \beta, t) \\ &+ \theta(|\xi| - |x|) D\left(\frac{x}{\xi}, t\right) \text{sgn}(\xi) \end{aligned} \quad (8)$$

guarantees the polynomiality property (4). Another interesting parametrization for GPDs supporting the polynomiality was suggested in Ref. [25].

The positivity bound on GPDs (7) is equivalent to the following representation for GPDs in the impact parameter representation (see the Appendix) in the region  $x > |\xi|$ :

$$\begin{aligned} \tilde{F}^{(N)}(x, \xi, b^\perp) &= (1-x)^{N+1} \sum_n \mathcal{Q}_n \left( \frac{1-x}{1+\xi}, (1-\xi)b^\perp \right) \\ &\times \mathcal{Q}_n \left( \frac{1-x}{1-\xi}, (1+\xi)b^\perp \right) \end{aligned} \quad (9)$$

with arbitrary real functions  $\mathcal{Q}_n$ . Instead of the discrete summation over  $n$  one can use the integration over continuous parameters.

Although both polynomiality and positivity are basic properties that must hold in any reasonable model of GPDs

usually the model building community meets a dilemma: one can use the double distribution representation (8) but it does not guarantee that the infinite set of inequalities (7) will be satisfied [26,27]. Alternatively one can build the models based on the representation (9) or on the so called overlap representation [15] which also automatically guarantees positivity bounds but then one meets problems with the polynomiality. In this paper a representation for GPDs is suggested which guarantees both positivity and polynomiality.

### III. MODIFIED DOUBLE DISTRIBUTION REPRESENTATION

For the construction of GPDs  $H^{(N)}(x, \xi, t)$  obeying both polynomiality and positivity constraints we use the double distribution representation which differs from the standard representation (8) by the extra factor of  $(1-x)^N$

$$\begin{aligned} H^{(N)}(x, \xi, t) &= (1-x)^N \int_{|\alpha|+|\beta|\leq 1} d\alpha d\beta \delta(x - \xi\alpha - \beta) \\ &\times F'_D(\alpha, \beta, t). \end{aligned} \quad (10)$$

Here  $N$  depends on the type of the parton distribution according to Table I.

Representation (10) obviously satisfies the polynomiality condition (4). Indeed,

$$\begin{aligned} \int_{-1}^1 dx x^n \int_{|\alpha|+|\beta|\leq 1} d\alpha d\beta \delta(x - \xi\alpha - \beta) F'_D(\alpha, \beta, t) \\ = P_n(\xi, t) \end{aligned} \quad (11)$$

where  $P_n(\xi, t)$  is a polynomial of degree  $n$ . Therefore

$$\begin{aligned} \int_{-1}^1 dx x^m H^{(N)}(x, \xi, t) &= \int_{-1}^1 dx \int_{|\alpha|+|\beta|\leq 1} d\alpha d\beta x^m (1-x)^N \\ &\times \delta(x - \xi\alpha - \beta) F'_D(\alpha, \beta, t) \\ &= S_{N+m}(\xi, t) \end{aligned} \quad (12)$$

is a polynomial in  $\xi$  of degree  $N+m$  in agreement with Eq. (4).

For our aims it is convenient to use parameters

$$\alpha_1 = \frac{1}{2}(1 - \beta - \alpha), \quad \alpha_2 = \frac{1}{2}(1 - \beta + \alpha) \quad (13)$$

instead of  $\alpha, \beta$ . Actually it is  $\alpha_1$  and  $\alpha_2$  ( $\alpha_1 = 1 - x - y$ ,  $\alpha_2 = y$  in terms of variables  $x, y$  used in Refs. [2,7]) that appear as  $\alpha$  parameters in the perturbative diagrammatic justification of the double distribution representation. The modified double distribution expressed in terms of parameters  $\alpha_1, \alpha_2$  will be denoted as follows:

$$F_D(\alpha_1, \alpha_2, t) \equiv F'_D(\alpha, \beta, t). \quad (14)$$

After these changes the modified double distribution representation (10) takes the following form:

$$H^{(N)}(x, \xi, t) = 2(1-x)^N \int_0^1 d\alpha_1 \int_0^{1-\alpha_1} d\alpha_2 F_D(\alpha_1, \alpha_2, t) \times \delta[x - \xi(\alpha_2 - \alpha_1) - (1 - \alpha_1 - \alpha_2)]. \quad (15)$$

Here we use the triangle integration region in the  $\alpha_1, \alpha_2$  plane which corresponds to the constraint  $\beta > 0$  in terms of variables  $\alpha, \beta$ . Hence our GPD vanishes in the “antiquark” region (for brevity we use the word “quark” for any type of partons):

$$H^{(N)}(x, \xi, t) = 0 \quad \text{if } x < -|\xi|. \quad (16)$$

Therefore we must take care about the positivity constraints only in the “quark” region  $x > |\xi|$ . Once this pure quark GPD is constructed we can use the transformation  $x \rightarrow -x$  to build GPDs with appropriate properties in both quark and antiquark regions.

In the case  $N > 0$  one can add the  $D$ -term to the modified double distribution representation (15):

$$H^{(N)}(x, \xi, t) \xrightarrow{N>0} H^{(N)}(x, \xi, t) + x^{N-1} D\left(\frac{x}{\xi}, t\right) \times \theta\left(1 - \left|\frac{x}{\xi}\right|\right) \text{sgn}(\xi). \quad (17)$$

In principle, using the trick of Ref. [28], one can include the  $D$ -term into the double distribution. But if one is interested in a parametrization of GPDs obeying the positivity and polynomiality constraints, then the  $D$ -term is useful: it is localized in the region  $|x| < |\xi|$  and therefore it does not appear in the positivity condition (7). The polynomiality is obvious for the  $D$ -term. Thus by adding an arbitrary  $D$ -term we violate neither polynomiality nor positivity.

#### IV. ANSATZ FOR DOUBLE DISTRIBUTIONS

Now the problem is to find double distributions  $F_D(\alpha_1, \alpha_2, t)$  which lead to GPDs  $H^{(N)}(x, \xi, t)$  obeying the positivity constraint. We use the following ansatz for the modified double distributions (15):

$$F_D(\alpha_1, \alpha_2, t) = \int_0^\infty d\lambda \lambda \int d\nu \left( \frac{1}{\lambda \alpha_1 \alpha_2} - t \right)^{-\nu-1} \times L_\nu(\lambda \alpha_1, \lambda \alpha_2). \quad (18)$$

Our double distribution is parametrized by an infinite set of functions  $L_\nu(w_1, w_2)$  defined for  $w_1, w_2 \geq 0$  and depending on parameter  $\nu$ . We assume that for any  $\nu$  function  $L_\nu(w_1, w_2)$  corresponds to a positive definite quadratic form in  $w_1, w_2$ , i.e. for any function  $\phi(w)$

$$\int_0^\infty dw_1 \int_0^\infty dw_2 L_\nu(w_1, w_2) \phi(w_1) \phi^*(w_2) \geq 0. \quad (19)$$

This is equivalent to the existence of the following integral representation for  $L_\nu(w_1, w_2)$

$$L_\nu(w_1, w_2) = \int d\rho F_\nu(w_1, \rho) F_\nu^*(w_2, \rho) \quad (20)$$

or to its discrete series analog. Since we are interested in real and  $\xi$ -even GPDs, we must use real functions  $L_\nu$  and  $F_\nu$ .

The lower limit of the integral over  $\nu$  on the right-hand side (RHS) of Eq. (18) determines the asymptotics of  $F_D(\alpha_1, \alpha_2, t)$  at large  $|t|$ . If one integrates over positive  $\nu$ , then  $F_D \sim |t|^{-1}$ . Functions  $L_\nu$  appearing in Eq. (18) have the  $\nu$  dependent dimension, which is slightly awkward but simplifies the equations.

Since  $\lambda, \alpha_1, \alpha_2 \geq 0$  and  $t \leq 0$  the following factor appearing in Eq. (18) is always positive:

$$\frac{1}{\lambda \alpha_1 \alpha_2} - t > 0. \quad (21)$$

Below it will be shown that for any set of positive definite functions  $L_\nu(w_1, w_2)$  under the assumption that the integrals on the RHS of Eq. (18) are convergent, the resulting double distribution  $F_D(\alpha_1, \alpha_2, t)$  (18) leads to the GPD  $H^{(N)}(x, \xi, t)$  (15) which satisfies the positivity bound (7). This check of positivity will be done in Sec. VII but first we prefer to derive some useful relations.

#### V. EXPRESSION FOR GPDs

Let us derive the expressions for GPDs  $H^{(N)}(x, \xi, t)$  corresponding to the double distribution (18). First we insert ansatz (18) for the double distribution  $F_D(\alpha_1, \alpha_2, t)$  into representation (15) for GPD  $H^{(N)}(x, \xi, t)$

$$H^{(N)}(x, \xi, t) = 2(1-x)^N \int_0^1 d\alpha_1 \int_0^{1-\alpha_1} d\alpha_2 \times \delta[x - \xi(\alpha_2 - \alpha_1) - (1 - \alpha_1 - \alpha_2)] \times \int_0^\infty d\lambda \lambda \int d\nu \left( \frac{1}{\lambda \alpha_1 \alpha_2} - t \right)^{-\nu-1} \times L_\nu(\lambda \alpha_1, \lambda \alpha_2). \quad (22)$$

We can rewrite this as follows:

$$H^{(N)}(x, \xi, t) = 2(1-x)^N \int_0^\infty d\lambda \lambda \int_0^\infty d\alpha_1 \int_0^\infty d\alpha_2 \times \theta(1 - \alpha_1 - \alpha_2) \delta[x - \xi(\alpha_2 - \alpha_1) - (1 - \alpha_1 - \alpha_2)] \times \int d\nu \left( \frac{1}{\lambda \alpha_1 \alpha_2} - t \right)^{-\nu-1} L_\nu(\lambda \alpha_1, \lambda \alpha_2). \quad (23)$$

Let us introduce new integration variables

$$w_k = \lambda \alpha_k \quad (24)$$

instead of  $\alpha_k$  and integrate over  $\lambda$  using the delta function

$$\begin{aligned}
H^{(N)}(x, \xi, t) &= 2(1-x)^{N-1} \int_0^\infty dw_1 \int_0^\infty dw_2 \theta\left(x - \xi \frac{w_2 - w_1}{w_1 + w_2}\right) \\
&\times \int d\nu \left( \frac{1}{w_2} \frac{1+\xi}{1-x} + \frac{1}{w_1} \frac{1-\xi}{1-x} - t \right)^{-\nu-1} L_\nu(w_1, w_2). \quad (25)
\end{aligned}$$

The step function does not vanish in the region  $x > |\xi|$  so that the above expression simplifies as follows:

$$\begin{aligned}
H^{(N)}(x, \xi, t)|_{x > |\xi|} &= 2(1-x)^{N-1} \int_0^\infty dw_1 \int_0^\infty dw_2 \\
&\times \int d\nu \left( \frac{1}{w_2} \frac{1+\xi}{1-x} + \frac{1}{w_1} \frac{1-\xi}{1-x} - t \right)^{-\nu-1} \\
&\times L_\nu(w_1, w_2). \quad (26)
\end{aligned}$$

This representation can be rewritten in the following form:

$$\begin{aligned}
H^{(N)}(x, \xi, t)|_{x > |\xi|} &= 2(1-x)^{N-1} \int_0^\infty dw_1 \int_0^\infty dw_2 \\
&\times \int_0^\infty d\gamma e^{t\gamma} \int d\nu \frac{\gamma^\nu}{\Gamma(\nu+1)} L_\nu(w_1, w_2) \\
&\times \exp\left[-\gamma \left( \frac{1}{w_2} \frac{1+\xi}{1-x} + \frac{1}{w_1} \frac{1-\xi}{1-x} \right)\right]. \quad (27)
\end{aligned}$$

## VI. FORWARD DISTRIBUTION

In the forward limit  $\xi \rightarrow 0$ ,  $t \rightarrow 0$  we obtain from Eq. (26)

$$\begin{aligned}
f(x) &= H^{(N)}(x, 0, 0) \\
&= 2(1-x)^{N-1} \int_0^\infty dw_1 \int_0^\infty dw_2 \\
&\times \int d\nu \left( \frac{w_1 w_2 (1-x)}{w_1 + w_2} \right)^{\nu+1} L_\nu(w_1, w_2). \quad (28)
\end{aligned}$$

The positivity of forward parton distributions is a consequence of the general positivity bounds on GPDs which will

be established in the next section. On the other hand, we can see the positivity of the forward parton distribution  $f(x)$  directly from Eq. (28):

$$\begin{aligned}
&\int_0^\infty dw_1 \int_0^\infty dw_2 \left( \frac{w_1 w_2}{w_1 + w_2} \right)^{\nu+1} L_\nu(w_1, w_2) \\
&= \int_0^\infty dw_1 \int_0^\infty dw_2 \frac{1}{\Gamma(\nu+1)} \int_0^\infty d\tau \tau^\nu \\
&\times \exp\left(-\tau \frac{w_1 + w_2}{w_1 w_2}\right) L_\nu(w_1, w_2) \\
&= \frac{1}{\Gamma(\nu+1)} \int_0^\infty d\tau \tau^\nu \int_0^\infty dw_1 \int_0^\infty dw_2 \\
&\times L_\nu(w_1, w_2) \exp\left(-\frac{\tau}{w_1}\right) \exp\left(-\frac{\tau}{w_2}\right) \geq 0. \quad (29)
\end{aligned}$$

The positivity of the RHS follows from the inequality

$$\int_0^\infty dw_1 \int_0^\infty dw_2 L_\nu(w_1, w_2) \exp\left(-\frac{\tau}{w_1}\right) \exp\left(-\frac{\tau}{w_2}\right) \geq 0 \quad (30)$$

which is a consequence of the positivity (19) of the quadratic form  $L_\nu(w_1, w_2)$ .

## VII. PROOF OF POSITIVITY

Now we want to show that the modified double distribution (18) with positive definite functions  $L_\nu$  (19) generates GPD  $H^{(N)}(x, \xi, t)$  which satisfies the positivity bounds (7). For the positivity bounds we need the GPD in the impact parameter representation (6)

$$\begin{aligned}
\tilde{F}^{(N)}\left(x, \xi, \frac{1-x}{1-\xi^2} b^\perp\right) &= \int \frac{d^2 \Delta^\perp}{(2\pi)^2} \exp\left[i \frac{1-x}{1-\xi^2} (\Delta^\perp b^\perp)\right] \\
&\times H^{(N)}\left(x, \xi, -\frac{|\Delta^\perp|^2 + 4\xi^2 M^2}{1-\xi^2}\right). \quad (31)
\end{aligned}$$

Using representation (27) for the GPDs  $H^{(N)}(x, \xi, t)$ , we obtain

$$\begin{aligned}
\tilde{F}^{(N)}\left(x, \xi, \frac{1-x}{1-\xi^2} b^\perp\right) &= \int \frac{d^2 \Delta^\perp}{(2\pi)^2} \exp\left[i \frac{1-x}{1-\xi^2} (\Delta^\perp b^\perp)\right] 2(1-x)^{N-1} \int_0^\infty dw_1 \int_0^\infty dw_2 \int_0^\infty d\gamma \\
&\times \exp\left(-\gamma \frac{|\Delta^\perp|^2 + 4\xi^2 M^2}{1-\xi^2}\right) \int d\nu \frac{\gamma^\nu}{\Gamma(\nu+1)} \exp\left[-\gamma \left( \frac{1}{w_2} \frac{1+\xi}{1-x} + \frac{1}{w_1} \frac{1-\xi}{1-x} \right)\right] L_\nu(w_1, w_2). \quad (32)
\end{aligned}$$

Integrating over  $\Delta^\perp$ , introducing compact notation

$$r_1 = \frac{1-x}{1+\xi}, \quad r_2 = \frac{1-x}{1-\xi}. \quad (33)$$

(see the Appendix), and rescaling the integration variables  $w_k \rightarrow w_k r_k$ , we find

$$\begin{aligned} \tilde{F}^{(N)}\left(x, \xi, \frac{1-x}{1-\xi^2} b^\perp\right) &= \frac{1}{2\pi} \left(\frac{2r_1 r_2}{r_1 + r_2}\right)^{N+1} \int_0^\infty \frac{d\gamma}{\gamma} \int_0^\infty dw_1 \int_0^\infty dw_2 \exp\left[-\frac{r_1 r_2}{4\gamma} |b^\perp|^2 - \gamma \frac{M^2(r_1 - r_2)^2}{r_1 r_2}\right] \\ &\quad \times \int d\nu \frac{\gamma^\nu}{\Gamma(\nu+1)} \exp\left[-\frac{\gamma(w_1 + w_2)}{r_1 r_2 w_1 w_2}\right] L_\nu(r_1 w_1, r_2 w_2). \end{aligned} \quad (34)$$

Next we change the integration variable  $\gamma \rightarrow \gamma r_1 r_2$

$$\begin{aligned} \tilde{F}^{(N)}\left(x, \xi, \frac{1-x}{1-\xi^2} b^\perp\right) &= \frac{1}{2\pi} \left(\frac{2r_1 r_2}{r_1 + r_2}\right)^{N+1} \int_0^\infty \frac{d\gamma}{\gamma} \int_0^\infty dw_1 \\ &\quad \times \int_0^\infty dw_2 \exp\left[-\frac{1}{4\gamma} |b^\perp|^2 - \gamma M^2(r_1 - r_2)^2\right] \end{aligned} \quad (35)$$

$$\times \int d\nu \frac{(\gamma r_1 r_2)^\nu}{\Gamma(\nu+1)} \exp\left[-\frac{\gamma(w_1 + w_2)}{w_1 w_2}\right] L_\nu(r_1 w_1, r_2 w_2) \quad (36)$$

and use the representation

$$\exp[-\gamma(r_1 - r_2)^2 M^2] = \frac{1}{2M\sqrt{\pi\gamma}} \int_{-\infty}^\infty ds \exp\left[-\frac{s^2}{4\gamma M^2} + is(r_2 - r_1)\right]. \quad (37)$$

Then

$$\begin{aligned} \tilde{F}^{(N)}\left(x, \xi, \frac{1-x}{1-\xi^2} b^\perp\right) &= \frac{1}{4M\pi^{3/2}} \left(\frac{2r_1 r_2}{r_1 + r_2}\right)^{N+1} \int_0^\infty d\gamma \exp\left(-\frac{1}{4\gamma} |b^\perp|^2\right) \int_{-\infty}^\infty ds \exp\left(-\frac{s^2}{4\gamma M^2}\right) \\ &\quad \times \int_0^\infty dw_1 \int_0^\infty dw_2 \exp\left[is(r_2 - r_1) - \frac{\gamma(w_1 + w_2)}{w_1 w_2}\right] \int d\nu \gamma^{\nu-3/2} \frac{(r_1 r_2)^\nu}{\Gamma(\nu+1)} L_\nu(w_1 r_1, w_2 r_2). \end{aligned} \quad (38)$$

Now we turn to the positivity bound (7) written in the form of the integral over  $r_1, r_2$ —see Eq. (A8) in the Appendix. The left-hand side of this inequality is

$$\begin{aligned} &\int_0^1 dr_1 \int_0^1 dr_2 (r_1 + r_2)^{N+1} p^*(r_2) p(r_1) \tilde{F}^{(N)}\left(x, \xi, \frac{1-x}{1-\xi^2} b^\perp\right) \\ &= \frac{2^{N-1}}{M\pi^{3/2}} \int d\nu \int_0^\infty d\gamma \frac{\gamma^{\nu-3/2}}{\Gamma(\nu+1)} \exp\left(-\frac{1}{4\gamma} |b^\perp|^2\right) \int_{-\infty}^\infty ds \exp\left(-\frac{s^2}{4\gamma M^2}\right) \int_0^1 dr_1 \int_0^\infty dw_1 \int_0^1 dr_2 \int_0^\infty dw_2 L_\nu(w_1 r_1, w_2 r_2) \\ &\quad \times \left[ p(r_1) r_1^{N+\nu+1} \exp\left(-isr_1 - \frac{\gamma}{w_1}\right) \right] \left[ p(r_2) r_2^{N+\nu+1} \exp\left(-isr_2 - \frac{\gamma}{w_2}\right) \right]^*. \end{aligned} \quad (39)$$

Here we can rescale integration variables  $w_k \rightarrow w_k / r_k$ . Then

$$\begin{aligned} &\int_0^1 dr_1 \int_0^\infty dw_1 \int_0^1 dr_2 \int_0^\infty dw_2 L_\nu(w_1 r_1, w_2 r_2) \left[ p(r_1) r_1^{N+\nu+1} \exp\left(-isr_1 - \frac{\gamma}{w_1}\right) \right] \left[ p(r_2) r_2^{N+\nu+1} \exp\left(-isr_2 - \frac{\gamma}{w_2}\right) \right]^* \\ &= \int_0^\infty dw_1 \int_0^\infty dw_2 L_\nu(w_1, w_2) \left[ \int_0^1 dr_1 p(r_1) r_1^{N+\nu} \exp\left(-isr_1 - \frac{\gamma r_1}{w_1}\right) \right] \left[ \int_0^1 dr_2 p(r_2) r_2^{N+\nu} \exp\left(-isr_2 - \frac{\gamma r_2}{w_2}\right) \right]^* \\ &= \int_0^\infty dw_1 \int_0^\infty dw_2 L_\nu(w_1, w_2) \phi_\nu(w_1) \phi_\nu^*(w_2) \geq 0 \end{aligned} \quad (40)$$

where

$$\phi_\nu(w) = \int_0^1 dr p(r) r^{N+\nu} \exp\left(-isr - \frac{\gamma r}{w}\right). \quad (41)$$

The RHS of Eq. (40) is positive since  $L_\nu(w_1, w_2)$  is positive definite. Combining Eqs. (39) and (40), we complete the proof of the positivity bound (A8) for the GPD generated by the double distribution (18).

### VIII. CONCLUSIONS

In this paper we have shown that representation (18) for the double distributions [understood in the sense of Eq. (15)] generates GPDs (25) satisfying both polynomiality and positivity constraints. Our representation (18) for double distributions involves arbitrary positive definite quadratic forms  $L_\nu(w_1, w_2)$ . Functions  $L_\nu(w_1, w_2)$  parametrizing GPDs depend on the same amount of variables  $(w_1, w_2, \nu)$  as GPDs themselves  $(x, \xi, t)$ . This means that the class of solutions of the polynomiality and positivity constraints found in this paper is rather wide. On the other hand, this set of solutions is not complete. Indeed, our ansatz (18) does not depend on the mass of the hadron  $M$  whereas the positivity and polynomiality constraints are sensitive to  $M$ : although  $M$  appears neither in the polynomiality condition (4) for  $H^{(N)}$  nor in the positivity bound (7) for  $\tilde{F}^{(N)}$ , the relation (6) between  $H^{(N)}$  and  $\tilde{F}^{(N)}$  contains the hadron mass  $M$ . This means that the combined constraints of positivity and polynomiality are sensitive to the hadron mass  $M$ . Therefore the absence of the  $M$  dependence in our ansatz (18) should mean that there must exist other solutions of the polynomiality and positivity constraints and one has to try other methods in order to find the other solutions. In particular, in Ref. [29] the solutions of the positivity and polynomiality constraints are constructed in terms of triangle perturbative diagrams.

The parametrization of GPDs suggested here seems to be constructive for the model building: the positive definite functions  $L_\nu(w_1, w_2)$  can be easily generated by using Eq. (20). One should not forget about the possibility to add the  $D$ -term (17) which is not constrained by the polynomiality and positivity.

Certainly apart from the positivity and polynomiality there are other theoretical and phenomenological constraints on GPDs and it would be interesting whether representation (18) allows to construct viable models of GPDs.

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### APPENDIX: SOLUTION OF THE POSITIVITY BOUNDS

In this appendix we derive the solution (9) of the positivity bounds (7). First let us define variables  $r_1, r_2$  which can be used instead of  $x, \xi$ :

$$r_1 = \frac{1-x}{1+\xi}, \quad r_2 = \frac{1-x}{1-\xi}, \quad (A1)$$

$$\xi = \frac{r_2 - r_1}{r_2 + r_1}, \quad x = 1 - \frac{2r_1 r_2}{r_1 + r_2}, \quad (A2)$$

$$\frac{2dx d\xi}{(1-x)^3} = \frac{dr_1 dr_2}{r_1^2 r_2^2}. \quad (A3)$$

The region covered by the positivity bounds (7)

$$x > |\xi| \quad (A4)$$

is mapped to the square in the  $r_1, r_2$  plane

$$0 < r_1, r_2 < 1. \quad (A5)$$

Inequality (7) takes the following form in terms of integration variables  $r_1, r_2$  (we keep variables  $x, \xi$  in GPDs implying that they are functions of  $r_1, r_2$ ):

$$\int_0^1 \frac{dr_1}{r_1^2} \int_0^1 \frac{dr_2}{r_2^2} \left( \frac{r_1 + r_2}{r_1 r_2} \right)^{N+1} \times p^*(r_2) p(r_1) \tilde{F}^{(N)} \left( x, \xi, \frac{1-x}{1-\xi^2} b^\perp \right) \geq 0. \quad (A6)$$

Since function  $p$  is arbitrary we can replace it

$$p(r_1) \rightarrow r_1^{N+3} p(r_1) \quad (A7)$$

which leads us to the equivalent form of inequality (A6)

$$\int_0^1 dr_1 \int_0^1 dr_2 (r_1 + r_2)^{N+1} \times p^*(r_2) p(r_1) \tilde{F}^{(N)} \left( x, \xi, \frac{1-x}{1-\xi^2} b^\perp \right) \geq 0. \quad (A8)$$

Inequality (A6) means that function

$$\left( \frac{r_1 + r_2}{r_1 r_2} \right)^{N+1} \tilde{F}^{(N)} \left( x, \xi, \frac{1-x}{1-\xi^2} b^\perp \right) \quad (A9)$$

must be a positive definite quadratic form, i.e. it has the following representation

$$\begin{aligned} & \left( \frac{r_1 + r_2}{2r_1 r_2} \right)^{N+1} \tilde{F}^{(N)} \left( x, \xi, \frac{1-x}{1-\xi^2} b^\perp \right) \\ &= \sum_n R_n(r_1, b^\perp) R_n^*(r_2, b^\perp) \end{aligned} \quad (A10)$$

with some functions  $R_n$ . Turning back to the variables  $x, \xi$ , we find



$$\begin{aligned} \tilde{F}^{(N)}(x, \xi, b^\perp) = (1-x)^{N+1} \sum_n R_n \left( \frac{1-x}{1+\xi}, \frac{1-\xi^2}{1-x} b^\perp \right) \\ \times R_n^* \left( \frac{1-x}{1-\xi}, \frac{1-\xi^2}{1-x} b^\perp \right). \end{aligned} \quad (\text{A11})$$

In the case of real and  $\xi$ -even GPDs, functions  $R_n$  are real.

Introducing functions

$$Q_n(r, b^\perp) = R_n \left( r, \frac{1}{r} b^\perp \right) \quad (\text{A12})$$

we obtain representation (9) for  $\tilde{F}^{(N)}(x, \xi, b^\perp)$ .

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